

# On one-sided interval edge colorings of biregular bipartite graphs

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A proper edge  $t$ -coloring of a graph  $G$  is a coloring of edges of  $G$  with colors  $1, 2, \dots, t$  such that all colors are used, and no two adjacent edges receive the same color. The set of colors of edges incident with a vertex  $x$  is called a spectrum of  $x$ . An arbitrary nonempty subset of consecutive integers is called an interval. We say that a proper edge  $t$ -coloring of a graph  $G$  is interval in the vertex  $x$  if the spectrum of  $x$  is an interval. We say that a proper edge  $t$ -coloring  $\varphi$  of a graph  $G$  is interval on a subset  $R_0$  of vertices of  $G$ , if for an arbitrary  $x \in R_0$ ,  $\varphi$  is interval in  $x$ . We say that a subset  $R$  of vertices of  $G$  has an  $i$ -property if there is a proper edge  $t$ -coloring of  $G$  which is interval on  $R$ . If  $G$  is a graph, and a subset  $R$  of its vertices has an  $i$ -property, then the minimum value of  $t$  for which there is a proper edge  $t$ -coloring of  $G$  interval on  $R$  is denoted by  $w_R(G)$ .

In this paper, for some bipartite graphs, we estimate the value of this parameter in that cases when  $R$  coincides with the set of all vertices of one part of the graph.

Keywords: proper edge coloring; interval spectrum' biregular bipartite graph

We consider undirected, finite graphs without loops and multiple edges.  $V(G)$  and  $E(G)$  denote the sets of vertices and edges of a graph  $G$ , respectively. For any vertex  $x \in V(G)$ , we denote by  $N_G(x)$  the set of vertices of a graph  $G$  adjacent to  $x$ . The degree of a vertex  $x$  of a graph  $G$  is denoted by  $d_G(x)$ , the maximum degree of a vertex of  $G$  by  $\Delta(G)$ . For a graph  $G$  and an arbitrary subset  $V_0 \subseteq V(G)$ , we denote by  $G[V_0]$  the subgraph of  $G$  induced by the subset  $V_0$  of its vertices.

Using a notation  $G(X, Y, E)$  for a bipartite graph  $G$ , we mean that  $G$  has a bipartition  $(X, Y)$ , and  $E = E(G)$ .

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element  $p$  and the maximum element  $q$  is denoted by  $[p, q]$ .

A function  $\varphi : E(G) \rightarrow [1, t]$  is called a proper edge  $t$ -coloring of a graph  $G$ , if all colors are used, and no two adjacent edges receive the same color.

The minimum  $t \in \mathbb{N}$  for which there exists a proper edge  $t$ -coloring of a graph  $G$  is denoted by  $\chi'(G)$  [25].

For a graph  $G$  and any  $t \in [\chi'(G), |E(G)|]$ , we denote by  $\alpha(G, t)$  the set of all proper edge  $t$ -colorings of  $G$ . Let

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G, t).$$

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If  $G$  is a graph,  $x \in V(G)$ ,  $\varphi \in \alpha(G)$ , then let us set

$$S_G(x, \varphi) \equiv \{\varphi(e)/e \in E(G), e \text{ is incident with } x\}.$$

We say that  $\varphi \in \alpha(G)$  is persistent-interval in the vertex  $x_0 \in V(G)$  of the graph  $G$  iff  $S_G(x_0, \varphi) = [1, d_G(x_0)]$ . We say that  $\varphi \in \alpha(G)$  is persistent-interval on the set  $R_0 \subseteq V(G)$  iff  $\varphi$  is persistent-interval in  $\forall x \in R_0$ .

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We say that a subset  $R$  of vertices of a graph  $G$  has an  $i$ -property iff there exists  $\varphi \in \alpha(G)$  interval on  $R$ ; for a subset  $R \subseteq V(G)$  with an  $i$ -property, the minimum value of  $t$  warranting existence of  $\varphi \in \alpha(G, t)$  interval on  $R$  is denoted by  $w_R(G)$ .

Notice that the problem of deciding whether the set of all vertices of an arbitrary graph has an  $i$ -property is  $NP$ -complete [7, 2, 17]. Unfortunately, even for an arbitrary bipartite graph (in this case the interest is strengthened owing to the application of an  $i$ -property in timetablings [6, 17]) the problem keeps the complexity of a general case [3, 12, 24]. Some positive results were obtained for graphs of certain classes with numerical or structural restrictions [11, 13, 20, 17, 19, 9, 21, 22, 14, 15, 27, 28]. The examples of bipartite graphs whose sets of vertices have not an  $i$ -property are given in [6, 13, 16, 24].

The subject of this research is a parameter  $w_R(G)$  of a bipartite graph  $G = G(X, Y, E)$  in that case when  $R$  coincides with the set of all vertices of one part of  $G$  (the exact value of this parameter for an arbitrary bipartite graph is not known as yet). We obtain an upper bound of the parameter being discussed for biregular [3, 4, 2, 5, 23] bipartite graphs, and the exact values of it in the case of the complete bipartite graph  $K_{m,n}$  ( $m \in \mathbb{N}, n \in \mathbb{N}$ ) as well.

The terms and concepts that we do not define can be found in [26].

First we recall some known results.

**Theorem 1** [7, 8, 17] *If  $R$  is the set of all vertices of one part of an arbitrary bipartite graph  $G = G(X, Y, E)$ , then: 1) there exists  $\varphi \in \alpha(G, |E|)$  interval on  $R$ , 2) for  $\forall t \in [w_R(G), |E|]$ , there exists  $\psi_t \in \alpha(G, t)$  interval on  $R$ .*

**Theorem 2** [7, 8, 1] *Let  $G = G(X, Y, E)$  be a bipartite graph. If for  $\forall e = (x, y) \in E$ , where  $x \in X, y \in Y$ , the inequality  $d_G(y) \leq d_G(x)$  is true, then  $\exists \varphi \in \alpha(G, \Delta(G))$  persistent-interval on  $X$ .*

**Corollary 1** [7, 8, 1] *Let  $G = G(X, Y, E)$  be a bipartite graph. If  $\max_{y \in Y} d_G(y) \leq \min_{x \in X} d_G(x)$ , then  $\exists \varphi \in \alpha(G, \Delta(G))$  persistent-interval on  $X$ .*

**Remark 1** *Note that Corollary 1 follows from the result of [10].*

Let  $H = H(\mu, \nu)$  be a  $(0, 1)$ -matrix with  $\mu$  rows,  $\nu$  columns, and with elements  $h_{ij}$ ,  $1 \leq i \leq \mu$ ,  $1 \leq j \leq \nu$ . The  $i$ -th row of  $H$ ,  $i \in [1, \mu]$ , is called collected, iff  $h_{ip} = h_{iq} = 1$ ,  $t \in [p, q]$  imply  $h_{it} = 1$ , and the inequality  $\sum_{j=1}^{\nu} h_{ij} \geq 1$  is true. Similarly, the  $j$ -th column of  $H$ ,  $j \in [1, \nu]$ , is called collected, iff  $h_{pj} = h_{qj} = 1$ ,  $t \in [p, q]$  imply  $h_{tj} = 1$ , and the

inequality  $\sum_{i=1}^{\mu} h_{ij} \geq 1$  is true. If all rows and all columns of  $H$  are collected, then for  $i$ -th row of  $H$ ,  $i \in [1, \mu]$ , we define the number  $\varepsilon(i, H) \equiv \min\{j/h_{ij} = 1\}$ .

$H$  is called a collected matrix, iff all its rows and all its columns are collected,  $h_{11} = h_{\mu\nu} = 1$ , and  $\varepsilon(1, H) \leq \varepsilon(2, H) \leq \dots \leq \varepsilon(\mu, H)$ .

$H$  is called a  $b$ -regular matrix ( $b \in \mathbb{N}$ ), iff for  $\forall i \in [1, \mu]$ ,  $\sum_{j=1}^{\nu} h_{ij} = b$ .  $H$  is called a  $c$ -compressed matrix ( $c \in \mathbb{N}$ ), iff for  $\forall j \in [1, \nu]$ ,  $\sum_{i=1}^{\mu} h_{ij} \leq c$ .

**Lemma 1** [18] *If a collected  $n$ -regular ( $n \in \mathbb{N}$ ) matrix  $P = P(m, w)$  with elements  $p_{ij}$  ( $1 \leq i \leq m, 1 \leq j \leq w$ ) is  $n$ -compressed, then  $w \geq \lceil \frac{m}{n} \rceil \cdot n$ .*

**Proof.** We use induction on  $\lceil \frac{m}{n} \rceil$ .

If  $\lceil \frac{m}{n} \rceil = 1$ , the statement is trivial.

Now assume that  $\lceil \frac{m}{n} \rceil = \lambda_0 \geq 2$ , and the statement is true for all collected  $n'$ -regular  $n'$ -compressed matrixes  $P'(m', w')$  with  $\lceil \frac{m'}{n'} \rceil \leq \lambda_0 - 1$ .

First of all let us prove that  $\varepsilon(n+1, P) \geq n+1$ . Assume the contrary:  $\varepsilon(n+1, P) \leq n$ . Since  $P$  is a collected  $n$ -regular matrix, we obtain  $\sum_{i=1}^m p_{in} \geq \sum_{i=1}^{n+1} p_{in} \geq n+1$ , which is impossible because  $P(m, w)$  is an  $n$ -compressed matrix. This contradiction shows that  $\varepsilon(n+1, P) \geq n+1$ .

Now let us form a new matrix  $P'(m-n, w-(\varepsilon(n+1, P)-1))$  by deleting from the matrix  $P$  the elements  $p_{ij}$ , which satisfy at least one of the inequalities  $i \leq n, j \leq \varepsilon(n+1, P)-1$ .

It is not difficult to see that  $P'(m-n, w-(\varepsilon(n+1, P)-1))$  is a collected  $n$ -regular  $n$ -compressed matrix with  $\lceil \frac{m-n}{n} \rceil = \lambda_0 - 1$ . By the induction hypothesis, we have

$$w - (\varepsilon(n+1, P) - 1) \geq \left\lceil \frac{m-n}{n} \right\rceil \cdot n,$$

which means that

$$w \geq (\lambda_0 - 1)n + \varepsilon(n+1, P) - 1 \geq (\lambda_0 - 1)n + n = \lambda_0 n = \left\lceil \frac{m}{n} \right\rceil \cdot n.$$

Now, for arbitrary positive integers  $m, l, n, k$ , where  $m \geq n$  and  $ml = nk$ , let us define the class  $Bip(m, l, n, k)$  of biregular bipartite graphs:

$$Bip(m, l, n, k) \equiv \left\{ G = G(X, Y, E) \middle/ \begin{array}{l} |X| = m, |Y| = n, \text{ for } \forall x \in X, d_G(x) = l, \\ \text{for } \forall y \in Y, d_G(y) = k. \end{array} \right\}$$

**Remark 2** *Clearly, if  $G \in Bip(m, l, n, k)$ , then  $\chi'(G) = k$ .*

**Theorem 3** *If  $G = G(X, Y, E) \in Bip(m, l, n, k)$ , then  $w_Y(G) = k$ ,  $w_X(G) \leq l \cdot \lceil \frac{m}{l} \rceil$ .*

**Proof.** The equality follows from Remark 2. Let us prove the inequality.

Let  $X = \{x_1, \dots, x_m\}$ . For  $\forall r \in [1, \lceil \frac{m}{l} \rceil]$ , define  $X_r \equiv \{x_{(r-1)l+1}, \dots, x_{rl}\}$ . Define  $X_{1+\lceil \frac{m}{l} \rceil} \equiv X \setminus \left( \bigcup_{i=1}^{\lceil \frac{m}{l} \rceil} X_i \right)$ . For  $\forall r \in [1, \lceil \frac{m}{l} \rceil]$ , define  $Y_r \equiv \bigcup_{x \in X_r} N_G(x)$ . Define  $Y_{1+\lceil \frac{m}{l} \rceil} \equiv \bigcup_{x \in X_{1+\lceil \frac{m}{l} \rceil}} N_G(x)$ . For  $\forall r \in [1, \lceil \frac{m}{l} \rceil]$ , define  $G_r \equiv G[X_r \cup Y_r]$ .

Consider the sequence  $G_1, G_2, \dots, G_{\lceil \frac{m}{l} \rceil}$  of subgraphs of the graph  $G$ . From Corollary 1, we obtain that for  $\forall i \in [1, \lceil \frac{m}{l} \rceil]$ , there is  $\varphi_i \in \alpha(G_i, l)$  persistent-interval on  $X_i$ .

Clearly, for  $\forall e \in E(G)$ , there exists the unique  $\xi(e)$ , satisfying the conditions  $\xi(e) \in [1, \lceil \frac{m}{l} \rceil]$  and  $e \in E(G_{\xi(e)})$ .

Define a function  $\psi : E(G) \rightarrow [1, l \cdot \lceil \frac{m}{l} \rceil]$ . For an arbitrary  $e \in E(G)$ , set  $\psi(e) \equiv (\xi(e) - 1) \cdot l + \varphi_{\xi(e)}(e)$ .

It is not difficult to see that  $\psi \in \alpha(G, l \cdot \lceil \frac{m}{l} \rceil)$  and  $\psi$  is interval on  $X$ . Hence,  $w_X(G) \leq l \cdot \lceil \frac{m}{l} \rceil$ .

**Theorem 4** *Let  $R$  be the set of all vertices of one part of the complete bipartite graph  $G = K_{m,n}$ , where  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}$ . Then*

$$w_R(G) = (m + n - |R|) \cdot \left\lceil \frac{|R|}{m + n - |R|} \right\rceil.$$

**Proof.** Without loss of generality we can assume that  $G$  has a bipartition  $(X, Y)$ , where  $X = \{x_1, \dots, x_m\}$ ,  $Y = \{y_1, \dots, y_n\}$ , and  $m \geq n$ .

**Case 1**  $|R| = Y$ . In this case the statement follows from Theorem 3; thus  $w_Y(G) = m$ .

**Case 2**  $|R| = X$ .

The inequality  $w_X(G) \leq n \cdot \lceil \frac{m}{n} \rceil$  follows from Theorem 3. Let us prove that  $w_X(G) \geq n \cdot \lceil \frac{m}{n} \rceil$ .

Consider an arbitrary proper edge  $w_X(G)$ -coloring  $\varphi$  of the graph  $G$ , which is interval on  $X$ .

Clearly, without loss of generality, we can assume that

$$\min(S_G(x_1, \varphi)) \leq \min(S_G(x_2, \varphi)) \leq \dots \leq \min(S_G(x_m, \varphi)).$$

Let us define a  $(0, 1)$ -matrix  $P(m, w_X(G))$  with  $m$  rows,  $w_X(G)$  columns, and with elements  $p_{ij}$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq w_X(G)$ . For  $\forall i \in [1, m]$ , and for  $\forall j \in [1, w_X(G)]$ , set

$$p_{ij} = \begin{cases} 1, & \text{if } j \in S_G(x_i, \varphi) \\ 0, & \text{if } j \notin S_G(x_i, \varphi). \end{cases}$$

It is not difficult to see that  $P(m, w_X(G))$  is a collected  $n$ -regular  $n$ -compressed matrix. From Lemma 1, we obtain  $w_X(G) \geq n \cdot \lceil \frac{m}{n} \rceil$ .

From Theorems 1 and 3, taking into account the proof of Case 2 of Theorem 4, we also obtain

**Theorem 5** *If  $G \in \text{Bip}(m, l, n, k)$ , then*

1. for  $\forall t \in [l \cdot \lceil \frac{m}{l} \rceil, ml]$ , there exists  $\varphi_t \in \alpha(G, t)$  interval on  $X$ ,
2. for  $\forall t \in [k, nk]$ , there exists  $\psi_t \in \alpha(G, t)$  interval on  $Y$ .

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